LAGRANGIAN THEORY OF CONSTRAINED SYSTEMS: COSMOLOGICAL APPLICATION

Giampiero Esposito[1,2], **Gabriele Gionti**[3], **Giuseppe Marmo**[1,2,4]
and Cosimo Stornaiolo[1,2]

[1] *Istituto Nazionale di Fisica Nucleare*
Mostra d'Oltremare Padiglione 20, 80125 Napoli, Italy;
[2] *Dipartimento di Scienze Fisiche*
Mostra d'Oltremare Padiglione 19, 80125 Napoli, Italy;
[3] *Scuola Internazionale Superiore di Studi Avanzati*
Via Beirut 2-4, 34013 Trieste, Italy;
[4] *Departamento de Fisica Teorica*
Facultad de Ciencias Fisicas
Universidad Complutense, 28040 Madrid, Spain.

Summary. - Previous work in the literature has studied the Hamiltonian structure of an R^2 model of gravity with torsion in a closed Friedmann-Robertson-Walker universe. Within the framework of Dirac's theory, torsion is found to lead to a second-class primary constraint linear in the momenta and a second-class secondary constraint quadratic in the momenta.

This paper studies in detail the same problem at a Lagrangian level, i.e. working on the tangent bundle rather than on phase space. The corresponding analysis is motivated by a more general program, aiming to obtain a manifestly covariant, multisymplectic framework for the analysis of relativistic theories of gravitation regarded as constrained systems. After an application of the Gotay-Nester Lagrangian analysis, the paper deals with the generalized method, which has the advantage of being applicable to any system of differential equations in implicit form. Multiplication of the second-order Lagrange equations by a vector with zero eigenvalue for the Hessian matrix yields the so-called first-generation constraints.

Remarkably, in the cosmological model here considered, if Lagrange equations are studied using second-order formalism a second-generation constraint is found which is absent in first-order formalism. This happens since first- and second-order formalisms are inequivalent. There are, however, no *a priori* reasons for arguing that one of the two is incorrect. First- and second-generation constraints are used to derive physical predictions for the cosmological model.

PACS numbers: 04.20.Cv, 04.50, 04.60.+n, 98.80.Dr

1. - Introduction.

In recent years part of the theoretical-physics community has started appreciating that constrained systems may be also studied using a tangent-bundle formalism [1], rather than working on phase space. From the point of view of relativistic theories of gravitation, this approach appears relevant since it may lead to a fully covariant treatment of the gravitational field regarded as a constrained system, without having to use a 3+1 split of the Lorentzian space-time manifold.

Relying on previous work on cosmology in [2], we perform the analysis, in a closed Friedmann-Robertson-Walker (hereafter referred to as FRW) universe, of a model of gravity with non-vanishing torsion where the Lagrangian density is proportional to the square of the scalar curvature. Of course, this is only a toy model, but its analysis deserves careful consideration because if one tries to quantize general relativity (GR) or theories with non-vanishing torsion within the framework of perturbative renormalization, one finds that quantized GR is perturbatively non-renormalizable, and in both theories the effective action acquires terms quadratic in the full Riemann-curvature tensor, i.e. terms proportional to $R_{abcd}R^{abcd}$, $R_{ab}R^{ab}$, $\left(R^a{}_a\right)^2$. In the most general case, these theories, which are non-linear in the curvature, are either non-unitary or perturbatively non-renormalizable [3]. Moreover, they are much studied as gauge theories of gravitation [4-6], although the corresponding Cauchy problem at the classical level may be ill-posed. However, to improve our understanding of all difficulties and limits of the perturbative-renormalization program in quantum gravity, and to check the consistency or inconsistency of inflationary-universe models based on these non-linear theories, it appears essential to complete a detailed analysis of these models of gravity.

In the torsion-free case, a relevant cosmological application was studied, for example, in [7], where an action functional was considered which is purely quadratic in the trace of the Ricci tensor. The motivation for that quantum-cosmological analysis was to obtain a Wick-rotated path integral whose integrand does not blow up exponentially after suitable conformal rescalings of the 4-metric. As a first step of the program motivated so far, a similar model was proposed in [2] (see corrections in [8]), from a Hamiltonian point of view, by applying Dirac's theory of constrained Hamiltonian systems at the classical level. The resulting idealized cosmological model is a constrained system with second-class constraints, which arise, as shown below, by virtue of torsion. In the following part of this introductory paragraph, to help the reader, we summarize the Hamiltonian analysis appearing in [2].

In our FRW model, spatial homogeneity and isotropy imply that, in a coordinate frame, the only non-vanishing components of the torsion tensor are ${S_{10}}^1 = {S_{20}}^2 = {S_{30}}^3 = Q(t)$. Denoting by N the lapse function, by $a(t)$ the 3-sphere radius of the closed FRW metric, and defining $\alpha \equiv \log(a)$, $\frac{Q}{N} \equiv y$, $\tau \equiv \int N \, dt$, $' \equiv \frac{d}{d\tau}$, the action functional

$$I \equiv \kappa \int \left({R^a}_a \right)^2 \sqrt{-\det g} \, d^4x \quad , \tag{1.1}$$

after integration on the 3-sphere, takes the form $I = \int \widetilde{L} \, d\tau$, where

$$\widetilde{L} \equiv \mu e^{3\alpha} \left[e^{-2\alpha} + 2(\alpha')^2 - 12y\alpha' + 16y^2 + \alpha'' - 2y' \right]^2 \quad , \tag{1.2}$$

and μ is a proportionality constant, denoted by μ^0 in equation (2.13) of [2]. Note that the point Lagrangian (1.2) is obtained by inserting the FRW hypothesis into the field-theory Lagrangian of (1.1). By a direct calculation, one can check that this procedure is correct in the particular case of FRW cosmologies in general relativity [9]. For alternative theories, such a method was found to be correct in [7] in the FRW torsion-free case. We are thus using the assumptions about the FRW symmetry to reduce ourselves to the study of point Lagrangians without having to build the complete Hamiltonian treatment for a generic curvature-squared field theory with torsion.

Since the addition to the Lagrangian of a total derivative leads to an equivalent set of field equations, we use this property to cast the theory in Hamiltonian form. To eliminate the square of second-order derivatives appearing in $\widetilde{L}$ one is thus led to define [2,7,10]

$$L \equiv \widetilde{L} - \frac{d}{d\tau} \Big[(\alpha' - 2y) z \Big] \quad , \tag{1.3}$$

where $(\alpha' - 2y)$ is proportional to the trace of the extrinsic-curvature tensor of the 3-sphere, and z is obtained differentiating $\widetilde{L}$ with respect to the highest derivative, i.e. defining $z \equiv \frac{\partial \widetilde{L}}{\partial x''}$, where $x \equiv \alpha - 2 \int y \, d\tau$. Thus, setting $y \equiv u'$, the Lagrangian defined in equation (1.3) becomes (cf. equation (2.1))

$$L = 16z(u')^2 + 2z(\alpha')^2 - 12zu'\alpha' + 2u'z' - z'\alpha' + ze^{-2\alpha} - \frac{z^2}{4\mu}e^{-3\alpha} \quad . \tag{1.4}$$

For comments on this choice of variables, see again [2,7]. Hence, defining $p_\alpha \equiv \frac{\partial L}{\partial \alpha'}$, $p_u \equiv \frac{\partial L}{\partial u'}$, $p_z \equiv \frac{\partial L}{\partial z'}$ (cf. section 2), one finds the primary constraint $\phi_1 \approx 0$, where the weak-equality symbol $\approx$ denotes an equality which only holds on the constraint manifold [11-13], and

$$\phi_1 \equiv 2p_\alpha + p_u - 4zp_z \quad . \tag{1.5}$$

The corresponding effective Hamiltonian $\widetilde{H}$ on the whole phase space is given by $\widetilde{H} \equiv H_c + \gamma\phi_1$, where H_c, the Legendre transform of L, takes the form

$$H_c = -4zp_z^2 + \frac{p_u p_z}{2} + \frac{z^2 e^{-3\alpha}}{4\mu} - ze^{-2\alpha} \quad . \tag{1.6}$$

The constraint ϕ_1 is preserved in time by requiring that its Poisson bracket with $\widetilde{H}$, as defined in [11], should vanish: $\left\{\phi_1, \widetilde{H}\right\} \approx 0$. This leads to the secondary constraint

$$\phi_2 \equiv 16zp_z^2 - 2p_u p_z + \frac{7z^2}{2\mu}e^{-3\alpha} - 8ze^{-2\alpha} \quad . \tag{1.7}$$

The constraints ϕ_1 and ϕ_2 are second-class, since, after evaluating their Poisson bracket, one obtains a function which does not vanish when ϕ_1 and ϕ_2 are set to zero. Hence γ can be obtained as $\gamma = -\frac{\left\{\phi_2, H_c\right\}}{\left\{\phi_2, \phi_1\right\}}$ as in equations (2.24)-(2.25) of [2]. Note that, since ϕ_1 and ϕ_2 are second-class, we can define a new Poisson bracket, the so-called Dirac brackets, in which second-class constraints can be set strongly to zero [11-13], i.e. they behave as

Casimir functions for Dirac brackets. This implies that the canonical Hamiltonian H_c in equation (1.6) may be also written as

$$H_c = -2zp_z^2 - p_\alpha p_z + \frac{z^2 e^{-3\alpha}}{4\mu} - ze^{-2\alpha} \quad , \tag{1.8}$$

which formally coincides with the torsion-free result [7]. The effective Hamiltonian $\widetilde{H}$, however, is not a linear combination of constraints and hence does not vanish in general. One then finds the field equations (2.27)-(2.32) of [2].

After this introductory paragraph, we can summarize the plan of our paper as follows. Section 2 performs a Lagrangian analysis of our second-class constrained system. The kernel of the pre-symplectic two-form, its non-vertical part, the secondary constraint Φ_2, the second-order vector field solving the field equations and tangent to the constraint manifold are derived in detail. Section 3 presents instead a constraint analysis within the framework of the recently proposed generalized method. Results and open problems are presented in section 4.

2. - Gotay-Nester Lagrangian analysis.

The Lagrangian L of the model outlined in section 1 is more conveniently re-written for our purposes in the form

$$L = 16zv_u^2 + 2zv_\alpha^2 - 12zv_u v_\alpha + 2v_u v_z - v_z v_\alpha + ze^{-2\alpha} - \frac{z^2}{4\mu} e^{-3\alpha} \quad , \tag{2.1}$$

where α', u', z' have been replaced by the tangent-bundle fibre coordinates v_α, v_u, v_z respectively. The corresponding GN Lagrangian analysis [14] is as follows [15]. We first evaluate the Cartan one-form $\theta_L \equiv \frac{\partial L}{\partial v^i}\, dq^i$, and the pre-symplectic two-form $\omega_L \equiv d\theta_L$ [16]. Given a vector field Y belonging to the tangent bundle, and setting to zero the contraction $i_Y \omega_L$, one thus finds the kernel ker ω_L of the pre-symplectic two-form. Moreover, denoting by E_L the energy function, with corresponding one-form dE_L, the vanishing of the contraction $i_{\widetilde{Y}_C} dE_L$ defines the secondary constraint Φ_2, for $\widetilde{Y}_C$ belonging to the non-vertical part of ker ω_L. Denoting by Y an element of ker ω_L, this Lagrangian definition of constraints is clearly understood acting with i_Y on both sides of the Euler-Lagrange equations written in the form

$$i_\Gamma \omega_L + dE_L = 0 \quad ,$$

and using the identity $i_Y i_\Gamma \omega_L = -i_\Gamma i_Y \omega_L$. This yields the condition $0 = -i_Y dE_L$. Thus, unless $i_Y dE_L$ is identically vanishing, such calculation shows that $\Phi_2 \equiv i_{\widetilde{Y}_C} dE_L$ is actually the secondary constraint of the theory (section 4). This constraint is then preserved by requiring that its Lie derivative along the vector field Γ which solves the Lagrange field equations should vanish.

Indeed, from equation (2.1) one easily finds that the Cartan one-form and the pre-symplectic two-form are given by

$$\theta_L = \Big(4zv_\alpha - 12zv_u - v_z\Big)d\alpha + \Big(32zv_u - 12zv_\alpha + 2v_z\Big)du + \Big(2v_u - v_\alpha\Big)dz \quad , \tag{2.2}$$

$$\begin{aligned} \omega_L =& \Big[\Big(4v_\alpha - 12v_u\Big)dz + 4z\ dv_\alpha - 12z\ dv_u - dv_z\Big] \wedge d\alpha \\ &+ \Big[\Big(32v_u - 12v_\alpha\Big)dz - 12z\ dv_\alpha + 32z\ dv_u + 2dv_z\Big] \wedge du \\ &+ \Big[2dv_u - dv_\alpha\Big] \wedge dz \quad . \end{aligned} \tag{2.3}$$

Moreover, for a vector field Y of the tangent bundle, whose general decomposition is

$$\begin{aligned} Y =& Y_\alpha(q,v)\frac{\partial}{\partial \alpha} + Y_u(q,v)\frac{\partial}{\partial u} + Y_z(q,v)\frac{\partial}{\partial z} \\ &+ Y_{v_\alpha}(q,v)\frac{\partial}{\partial v_\alpha} + Y_{v_u}(q,v)\frac{\partial}{\partial v_u} + Y_{v_z}(q,v)\frac{\partial}{\partial v_z} \quad , \end{aligned} \tag{2.4}$$

the contraction with ω_L is evaluated according to the formula

$$i_Y \omega_L = \frac{\partial^2 L}{\partial v^i \partial v^j}\Big(Y_v^j\ dq^i - Y_q^i\ dv^j\Big) + \frac{\partial^2 L}{\partial v^i \partial q^j}\Big(Y_q^j\ dq^i - Y_q^i\ dq^j\Big) \quad . \tag{2.5}$$

After a lengthy calculation, this yields

$$
\begin{aligned}
i_Y\omega_L =& \Big[4zY_{v_\alpha} - 12zY_{v_u} - Y_{v_z} + \Big(4v_\alpha - 12v_u\Big)Y_z\Big]\,d\alpha \\
&+ \Big[-12zY_{v_\alpha} + 32zY_{v_u} + 2Y_{v_z} + \Big(32v_u - 12v_\alpha\Big)Y_z\Big]\,du \\
&+ \Big[-Y_{v_\alpha} + 2Y_{v_u} + \Big(12v_u - 4v_\alpha\Big)Y_\alpha + \Big(12v_\alpha - 32v_u\Big)Y_u\Big]\,dz \\
&+ \Big[-4zY_\alpha + 12zY_u + Y_z\Big]\,dv_\alpha + \Big[12zY_\alpha - 32zY_u - 2Y_z\Big]\,dv_u \\
&+ \Big(Y_\alpha - 2Y_u\Big)\,dv_z \quad .
\end{aligned}
\tag{2.6}
$$

Thus, if we set to zero $i_Y\omega_L$, the vector field $\widetilde{Y} \in \ker \omega_L$ is found to take the form

$$
\begin{aligned}
\widetilde{Y} =& Y_\alpha \frac{\partial}{\partial \alpha} + \frac{1}{2}Y_\alpha \frac{\partial}{\partial u} - 2zY_\alpha \frac{\partial}{\partial z} + Y_{v_\alpha}\frac{\partial}{\partial v_\alpha} + \Big[\frac{1}{2}Y_{v_\alpha} + \Big(2v_u - v_\alpha\Big)Y_\alpha\Big]\frac{\partial}{\partial v_u} \\
&+ \Big[-2zY_{v_\alpha} + 4zv_\alpha Y_\alpha\Big]\frac{\partial}{\partial v_z} \quad .
\end{aligned}
\tag{2.7}
$$

Note that, in equation (2.7), Y_α and Y_{v_α} remain arbitrary functions of $\alpha, u, z, v_\alpha, v_u$ and v_z. For $Y_\alpha = 0$ we get the vertical kernel. For them $dE_L(\widetilde{Y}) = 0$ is identically satisfied. By setting $Y_{v_\alpha} = 0$ we get vector fields $\widetilde{Y}_C$ giving rise to constraints. Thus, since the energy function $E_L \equiv v^i \frac{\partial L}{\partial v^i} - L$ is, in our case, such that

$$
E_L = 16zv_u^2 + 2zv_\alpha^2 - 12zv_uv_\alpha + 2v_uv_z - v_zv_\alpha + \frac{z^2}{4\mu}e^{-3\alpha} - ze^{-2\alpha} \quad , \tag{2.8}
$$

$$
\begin{aligned}
dE_L =& \Big(2ze^{-2\alpha} - \frac{3z^2}{4\mu}e^{-3\alpha}\Big)d\alpha + \Big(2v_\alpha^2 - 12v_uv_\alpha + 16v_u^2 - e^{-2\alpha} + \frac{z}{2\mu}e^{-3\alpha}\Big)dz \\
&+ \Big(4zv_\alpha - 12zv_u - v_z\Big)dv_\alpha + \Big(-12zv_\alpha + 32zv_u + 2v_z\Big)dv_u \\
&+ \Big(2v_u - v_\alpha\Big)dv_z \quad ,
\end{aligned}
\tag{2.9}
$$

the secondary constraint Φ_2 can be found as

$$
i_{\widetilde{Y}_C} dE_L \equiv \Phi_2 = \Big(2v_u - v_\alpha\Big)\Big(16zv_u - 4zv_\alpha + 2v_z\Big) + z\Big(4e^{-2\alpha} - \frac{7z}{4\mu}e^{-3\alpha}\Big) \quad . \tag{2.10}
$$

Let us now consider a vector field Γ which solves the Lagrange field equations

$$i_\Gamma \omega_L = -dE_L \quad . \tag{2.11}$$

In light of equations (2.6) and (2.9), equation (2.11) yields by comparison

$$\Gamma_u = \left(v_u - \frac{v_\alpha}{2}\right) + \frac{1}{2}\Gamma_\alpha \quad , \tag{2.12}$$

$$\Gamma_z = 2zv_\alpha + v_z - 2z\Gamma_\alpha \quad , \tag{2.13}$$

$$\Gamma_{v_u} = \frac{1}{2}\Gamma_{v_\alpha} + \left(2v_u - v_\alpha\right)\Gamma_\alpha + \left(2v_\alpha + \frac{v_z}{z}\right)\left(\frac{v_\alpha}{2} - v_u\right) - \frac{1}{16}\left(8e^{-2\alpha} - \frac{3z}{\mu}e^{-3\alpha}\right) \quad , \tag{2.14}$$

$$\Gamma_{v_z} = -2z\Gamma_{v_\alpha} + 4zv_\alpha\Gamma_\alpha - 2v_\alpha\left(2zv_\alpha + v_z\right) + z\left(8e^{-2\alpha} - \frac{3z}{\mu}e^{-3\alpha}\right) \quad , \tag{2.15}$$

whereas Γ_α and Γ_{v_α} remain arbitrary functions of α, u, z, v_α, v_u and v_z. The dynamics is tangent to the constraint manifold if the Lie derivative along Γ of the secondary constraint Φ_2 in equation (2.10) is vanishing, i.e. $\mathcal{L}_\Gamma \Phi_2 = 0$. This means that the contraction $i_\Gamma d\Phi_2$ should vanish, and leads to a restriction on the coefficient Γ_α, which is found to take the value $\widetilde{\Gamma}_\alpha$ such that

$$\widetilde{\Gamma}_\alpha = \frac{\left(2zv_\alpha + v_z\right)\left[\frac{11z}{4\mu}e^{-3\alpha} - 2e^{-2\alpha} + 2\left(2v_u - v_\alpha\right)\left(8v_u - 2v_\alpha + \frac{v_z}{z}\right)\right]}{\left[\frac{49z^2}{4\mu}e^{-3\alpha} - 16ze^{-2\alpha} + 4z\left(2v_u - v_\alpha\right)\left(8v_u - 2v_\alpha + \frac{v_z}{z}\right)\right]} \quad , \tag{2.16}$$

since the one-form $d\Phi_2$ is

$$\begin{aligned} d\Phi_2 = & \left(\frac{21z^2}{4\mu}e^{-3\alpha} - 8ze^{-2\alpha}\right)d\alpha + \left(4v_\alpha^2 - 24v_u v_\alpha + 32v_u^2 + 4e^{-2\alpha} - \frac{7z}{2\mu}e^{-3\alpha}\right)dz \\ & + \left(8zv_\alpha - 24zv_u - 2v_z\right)dv_\alpha + \left(-24zv_\alpha + 64zv_u + 4v_z\right)dv_u \\ & + \left(4v_u - 2v_\alpha\right)dv_z \quad , \end{aligned} \tag{2.17}$$

and the various coefficients of Γ_{v_α} appearing in $i_\Gamma d\Phi_2$ add up to zero. The geometrical meaning of our calculation is as follows. If Γ solves the Lagrange equations (2.11), for any vector field $\widetilde{Y} \in \ker \omega_L$ one finds

$$i_{\Gamma+\widetilde{Y}}\, \omega_L = \Bigl(i_\Gamma\, \omega_L\Bigr) + \Bigl(i_{\widetilde{Y}}\, \omega_L\Bigr) = -dE_L \quad . \tag{2.18}$$

In other words, the arbitrariness of Γ_α and Γ_{v_α} reflects the possibility of adding to any solution of equation (2.11) an arbitrary vector field $\widetilde{Y} \in \ker \omega_L$. However, if the vector field Γ is also tangent to the constraint manifold, only Γ_{v_α} remains arbitrary. The vector field Γ_T that solves equation (2.11) and is tangent to the constraint manifold is thus found to be

$$\Gamma_T = \widetilde{\Gamma}_\alpha \frac{\partial}{\partial\alpha} + \widetilde{\Gamma}_u \frac{\partial}{\partial u} + \widetilde{\Gamma}_z \frac{\partial}{\partial z} + \Gamma_{v_\alpha} \frac{\partial}{\partial v_\alpha} + \widetilde{\Gamma}_{v_u} \frac{\partial}{\partial v_u} + \widetilde{\Gamma}_{v_z} \frac{\partial}{\partial v_z} \quad , \tag{2.19}$$

where $\widetilde{\Gamma}_\alpha$ has been evaluated as in equation (2.16), and $\widetilde{\Gamma}_u$, $\widetilde{\Gamma}_z$, $\widetilde{\Gamma}_{v_u}$, $\widetilde{\Gamma}_{v_z}$ are values taken by the right-hand sides of equations (2.12)-(2.15) at $\Gamma_\alpha = \widetilde{\Gamma}_\alpha$. The arbitrariness of Γ_{v_α} is due to the existence of a vertical kernel.

3. - Generalized method.

We here study a different and more recent method for the constrained analysis of a dynamical system. Interestingly, it does not rely on a Cartan one-form or a pre-symplectic two-form as the method studied in section 2, but enables one to derive constraints by looking directly at the equations of motion, and can be applied to any system of differential equations in implicit form.

To describe the method, let us assume that a system of implicit dynamical equations is given in the particular form

$$a_{ij}(q,v)\dot{v}^i - f_j(q,v) = 0 \quad , \tag{3.1}$$

$$b_{ij}(q,v)\Bigl(\dot{q}^i - v^i\Bigr) = 0 \quad . \tag{3.2}$$

One now has a choice of first- or second-order formalism. If first-order theory is used, the time-derivatives of the positions q^i are not identified with the velocities v^i. Denoting by ψ^j a vector such that $\psi^j b_{ij} = 0$, the most general form of Eq. (3.2) is

$$b_{ij}(q,v)\Big(\dot{q}^i - v^i + \psi^i\Big) = 0 \quad . \tag{3.3}$$

Thus, in first-order theory, one finds

$$\dot{q}^i = v^i - \psi^i \quad . \tag{3.4}$$

The constraints are found by solving for ϕ^j the equation $\phi^j a_{ij} = 0$. This leads to a compatibility condition of the kind $\phi^j f_j = 0$. If this equation is not identically satisfied then $\phi^j f_j$ is the first-generation constraint of the theory. The constraints are preserved by requiring that

$$\frac{d}{dt}\Big(\phi^j f_j\Big) = \frac{\partial\Big(\phi^j f_j\Big)}{\partial q^k}\dot{q}^k + \frac{\partial\Big(\phi^j f_j\Big)}{\partial v^k}\dot{v}^k = 0 \quad . \tag{3.5}$$

In second-order theory one has instead $\dot{q}^i = v^i$, hence $\psi^i = 0$ in (3.4). Note that the arguments developed so far do not rely on the existence of a Lagrangian, in agreement with what we said.

However, if a Lagrangian is known for the model under consideration, a_{ij} is the Hessian matrix H_{ij}, whereas $b_{ij} = \frac{\partial^2 L}{\partial q^i \partial \dot{q}^j} - \frac{\partial^2 L}{\partial q^j \partial \dot{q}^i}$. Multiplying by a vector A^i the second-order Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial v^i} - \frac{\partial L}{\partial q^i} = 0 \quad , \tag{3.6}$$

where

$$v^i(t) = \frac{d}{dt}q^i(t) \quad , \tag{3.7}$$

one finds

$$\frac{\partial^2 L}{\partial v^i \partial q^j}v^j A^i - \frac{\partial L}{\partial q^i}A^i = 0 \quad , \tag{3.8}$$

which is a restriction on Cauchy data, i.e. a constraint. Following [17-18], such constraints are called *first-generation* constraints. Clearly, the number of independent eigenvectors corresponding to zero eigenvalues of H_{ij} is $N-K$, where K is the maximal rank of the Hessian matrix. Hence first-generation constraints can be denoted by $\psi_m^{\widehat{\mathrm{I}}}$, $m = 1, 2, ..., M \leq N-K$. Second-generation constraints (if any) are then obtained by requiring that the evolution of the system should be tangent to the first-generation constraint manifold, i.e.

$$\frac{d}{dt}\psi^{\widehat{\mathrm{I}}} \equiv \psi^{\widehat{\mathrm{II}}} = 0 \quad . \tag{3.9}$$

Analogously, third-generation constraints are determined by imposing that the evolution should be tangent to the manifold defined by first- and second-generation constraints, and so on.

A similar analysis can be repeated for first-order Lagrange equations. This leads to

$$\frac{\partial^2 L}{\partial v^i \partial v^j} v^j A_q^i + \left(\frac{\partial L}{\partial v^i} - \frac{\partial^2 L}{\partial q^i \partial v^j} v^j \right) A_v^i = 0 \quad . \tag{3.10}$$

A first-generation constraint has been thus obtained within the first-order formalism. All further constraints (i.e. second-generation, third-generation and so on) are then found by imposing that the evolution of the system should be tangent to the constraint manifold.

We now apply this new method to our cosmological model. The first-order equations of motion are

$$2\Bigl(\dot{u} - v_u\Bigr) - \Bigl(\dot{\alpha} - v_\alpha\Bigr) = 0 \quad , \tag{3.11}$$

$$2z\Bigl(\dot{z} - v_z\Bigr) - 12z\Bigl(\dot{\alpha} - v_\alpha\Bigr) + 32z\Bigl(\dot{u} - v_u\Bigr) = 0 \quad , \tag{3.12}$$

$$4z\Bigl(\dot{\alpha} - v_\alpha\Bigr) - 12z\Bigl(\dot{u} - v_u\Bigr) - \Bigl(\dot{z} - v_z\Bigr) = 0 \quad , \tag{3.13}$$

$$\dot{v}_z - 6\dot{z}v_\alpha - 6z\dot{v}_\alpha + 16\dot{z}v_u + 16\dot{v}_u z = 0 \quad , \tag{3.14}$$

$$-\dot{v}_z + 4\dot{z}v_\alpha + 4z\dot{v}_\alpha - 12z\dot{v}_u - 12\dot{z}v_u + 2ze^{-2\alpha} - \frac{3}{4}\frac{z^2}{\mu}e^{-3\alpha} = 0 \quad , \tag{3.15}$$

$$
\begin{aligned}
-\dot{v}_\alpha + 2\dot{v}_u - e^{-2\alpha} - 2v_\alpha^2 + 12v_u v_\alpha - 16v_u^2 + \frac{z}{2\mu}e^{-3\alpha} \\
= \Big(4v_\alpha - 12v_u\Big)\Big(\dot{\alpha} - v_\alpha\Big) \\
+ \Big(32v_u - 12v_\alpha\Big)\Big(\dot{u} - v_u\Big) \quad . \qquad (3.16)
\end{aligned}
$$

Since the Hessian matrix H_{ij} is of rank 2 in our case, its kernel is one-dimensional. The vector of the kernel of H_{ij} is found to be

$$
A^i = \begin{pmatrix} 2 \\ 1 \\ -4z \end{pmatrix} \quad . \qquad (3.17)
$$

The equations of motion (3.14)-(3.16) can be seen as a row vector which, multiplied by A^i, yields the first-generation constraint. Such a constraint is thus defined by

$$
\psi^{\widehat{\mathrm{I}}} \equiv A^i \frac{\partial^2 L}{\partial v^i \partial q^j}\dot{q}^j - A^i \frac{\partial L}{\partial q^i} - A^i \frac{\partial^2 L}{\partial v^j \partial q^i}\Big(\dot{q}^j - v^j\Big) = 0 \quad . \qquad (3.18)
$$

Bearing in mind equations (3.11)-(3.13), one finds

$$
\psi^{\widehat{\mathrm{I}}} = \Big(2v_u - v_\alpha\Big)\Big(8zv_u - 2zv_\alpha + v_z\Big) + z\left(2e^{-2\alpha} - \frac{7z}{8\mu}e^{-3\alpha}\right) \quad . \qquad (3.19)
$$

The constraint $\psi^{\widehat{\mathrm{I}}}$ is preserved if $\frac{d}{dt}\psi^{\widehat{\mathrm{I}}} = 0$, which implies

$$
\dot{\alpha} = \frac{\Big(2zv_\alpha + v_z\Big)\left[\frac{11z}{4\mu}e^{-3\alpha} - 2e^{-2\alpha} + 2\Big(2v_u - v_\alpha\Big)\Big(8v_u - 2v_\alpha + \frac{v_z}{z}\Big)\right]}{\left[\frac{49z^2}{4\mu}e^{-3\alpha} - 16ze^{-2\alpha} + 4z\Big(2v_u - v_\alpha\Big)\Big(8v_u - 2v_\alpha + \frac{v_z}{z}\Big)\right]} \quad . \qquad (3.20)
$$

By imposing that the evolution of the system should be tangent to the constraint manifold we have derived a formula for $\dot{\alpha}$, which is the analogue of $\widetilde{\Gamma}_\alpha$ in the Gotay-Nester analysis of section 2 (Eq. (2.16)). Thus, by using the generalized method, the constraint analysis is completed once we determine $\dot{\alpha}$, which remains arbitrary if one looks simply at the equations of motion which govern the dynamics.

It is now very instructive to perform the constraint analysis within the second-order formalism. The second-order Lagrange equations (3.6)-(3.7) read

$$v_\alpha = \dot{\alpha} \quad , \tag{3.21}$$

$$v_u = \dot{u} \quad , \tag{3.22}$$

$$v_z = \dot{z} \quad , \tag{3.23}$$

$$\dot{v}_z - 6v_z v_\alpha - 6z\dot{v}_\alpha + 16v_z v_u + 16\dot{v}_u z = 0 \quad , \tag{3.24}$$

$$-\dot{v}_z + 4v_z v_\alpha + 4z\dot{v}_\alpha - 12z\dot{v}_u - 12v_z v_u + 2ze^{-2\alpha} - \frac{3}{4}\frac{z^2}{\mu}e^{-3\alpha} = 0 \quad , \tag{3.25}$$

$$-\dot{v}_\alpha + 2\dot{v}_u - e^{-2\alpha} - 2v_\alpha^2 + 12v_u v_\alpha - 16v_u^2 + \frac{z}{2\mu}e^{-3\alpha} = 0 \quad . \tag{3.26}$$

The kernel of the Hessian matrix is again given by (3.17), and the first-generation constraint (3.8) turns out to coincide with the first-generation constraint (3.19). The search for second-generation constraints leads to

$$\begin{aligned}\psi^{\widehat{\mathrm{II}}} &\equiv \frac{d}{dt}\psi^{\widehat{\mathrm{I}}} \\ &= \dot{\alpha}\left[\frac{49z^2}{4\mu}e^{-3\alpha} - 16ze^{-2\alpha} + 4z\Big(2\dot{u} - \dot{\alpha}\Big)\Big(8\dot{u} - 2\dot{\alpha} + \frac{\dot{z}}{z}\Big)\right] \\ &-\Big(2z\dot{\alpha} + \dot{z}\Big)\left[\frac{11z}{4\mu}e^{-3\alpha} - 2e^{-2\alpha} + 2\Big(2\dot{u} - \dot{\alpha}\Big)\Big(8\dot{u} - 2\dot{\alpha} + \frac{\dot{z}}{z}\Big)\right] \quad .\end{aligned} \tag{3.27}$$

Requiring that the evolution of the system should be tangent to the constraint manifold, and defining

$$\widetilde{D} \equiv 16ze^{-2\alpha} - \frac{49}{4\mu}z^2 e^{-3\alpha} + 4z\Big(\dot{\alpha} - 2\dot{u}\Big)\Big(8\dot{u} - 2\dot{\alpha} + \frac{\dot{z}}{z}\Big) \quad , \tag{3.28}$$

one finds

$$\begin{aligned}
\widetilde{D}\dot{\alpha} = \dot{\alpha}\Bigg[&\frac{e^{-3\alpha}}{4\mu}\Big(-123z^2\dot{\alpha}+98z\dot{z}-48z^2\dot{u}\Big)\\
&+e^{-2\alpha}\Big(16z\dot{\alpha}-16\dot{z}+32z\dot{u}\Big)\\
&-\frac{z}{2}\Big(8\dot{u}-2\dot{\alpha}+\frac{\dot{z}}{z}\Big)\Big(8e^{-2\alpha}-\frac{3z}{\mu}e^{-3\alpha}\Big)\\
&+4\Big(2\dot{u}-\dot{\alpha}\Big)\Big(\Big(2\dot{\alpha}-8\dot{u}\Big)\dot{z}-\frac{\dot{z}^2}{z}\Big)\Bigg]\\
&-z\Big(8e^{-2\alpha}-\frac{3z}{\mu}e^{-3\alpha}\Big)\Big[\frac{11z}{4\mu}e^{-3\alpha}-2e^{-2\alpha}+2\Big(2\dot{u}-\dot{\alpha}\Big)\Big(8\dot{u}-2\dot{\alpha}+\frac{\dot{z}}{z}\Big)\Big]\\
&-4\Big(2z\dot{\alpha}+\dot{z}\Big)\Big(\dot{\alpha}-2\dot{u}\Big)\Big(8\dot{u}-2\dot{\alpha}+\frac{\dot{z}}{z}\Big)\\
&+\frac{e^{-3\alpha}}{4\mu}\Big(2z\dot{\alpha}+\dot{z}\Big)\Big(27z\dot{\alpha}-14\dot{z}\Big)\\
&+2e^{-2\alpha}\Big(2z\dot{\alpha}+\dot{z}\Big)\frac{\dot{z}}{z} \quad . \qquad (3.29)
\end{aligned}$$

Note that a substantial difference occurs with respect to a first-order formalism, since we find a second-generation constraint which is absent in the first-order case. Hence the variable v_α is determined, which remains instead arbitrary in the first-order analysis.

The occurrence of the second-generation constraint, which has no equivalent in Dirac's Hamiltonian treatment [2] of section 1, is not a peculiar property of the *generalized method* of this section, but rather of the second-order formalism. A second-order Gotay-Nester analysis yields the same result. Indeed, for any point Lagrangian in a second-order theory with constraints, the contraction of the energy one-form with a vector field A in the kernel of the pre-symplectic two-form yields the constraint

$$\phi(q,\dot{q}) \equiv A^s\left[\dot{q}^h\frac{\partial^2 L}{\partial\dot{q}^s\partial q^h}-\frac{\partial L}{\partial q^s}\right] \quad , \qquad (3.30)$$

which coincides with the second-order first-generation constraint (3.7)-(3.8). The Lie derivative of $\phi(q,\dot{q})$ along the second-order vector field

$$\Gamma = \dot{q}^i \frac{\partial}{\partial q^i} + \Gamma^i_{\dot{q}} \frac{\partial}{\partial \dot{q}^i} \quad , \tag{3.31}$$

leads to a further constraint which, in our paper, coincides (by construction) with the second-generation constraint (3.27).

Indeed, our model is not the only case of point Lagrangian where second-order formalism leads to further constraints with respect to first-order formalism. For example, if one studies the point Lagrangian (cf. [19])

$$L \equiv \frac{1}{2}v_1^2 + v_1 q_2 + \Big(1-\alpha\Big) v_2 q_1 + \frac{\beta}{2}\Big(q_1 - q_2\Big)^2 \quad , \tag{3.32}$$

with $\alpha^2 - \beta \neq 0$, first-order formalism only leads to the constraint

$$\Phi_2 \equiv -\alpha v_1 + \beta\Big(q_1 - q_2\Big) \quad , \tag{3.33}$$

whereas second-order formalism also leads to the further constraint

$$\Phi_3 \equiv \Big(\alpha^2 - \beta\Big)\dot{q}_2 + \beta\dot{q}_1 - \alpha\beta\Big(q_1 - q_2\Big) \quad . \tag{3.34}$$

In the latter case, the vector field solving the Lagrange field equations (2.11) takes the form

$$\Gamma = \dot{q}_1 \frac{\partial}{\partial q_1} + \dot{q}_2 \frac{\partial}{\partial q_2} + \bigg(-\alpha\dot{q}_2 + \beta\Big(q_1 - q_2\Big)\bigg) \frac{\partial}{\partial \dot{q}_1} \quad . \tag{3.35}$$

The additional constraint (3.34) is obtained since velocities have been taken to be time-derivatives of positions.

A further example of the inequivalence of first- and second-order theory is given by the following Lagrangian on the tangent bundle of $SU(2)$:

$$L \equiv \mathrm{Tr}\Big(\sigma_3 S^{-1}\dot{S}\Big) \quad , \tag{3.36}$$

where $S \in SU(2)$. The corresponding pre-symplectic 2-form is found to be

$$\omega_L = -\text{Tr}\ \sigma_3 \Big(S^{-1}dS \wedge S^{-1}dS\Big) \quad , \tag{3.37}$$

and the first-order dynamics is given by the vector field X_3 associated with the one-parameter subgroup $e^{\frac{it}{2}\sigma_3}$. In second-order formalism, however, there is *no dynamics* compatible with the Lagrangian (3.36). This Lagrangian is a Chern-Simons Term, and hence is relevant for modern field theory.

Interestingly, we may use the constraints (3.19) and (3.27) to derive physical predictions. In other words, by imposing the first-generation constraint $\psi^{\widehat{\mathrm{I}}} = 0$ we may cast the second-generation constraint in the form

$$\psi^{\widehat{\mathrm{II}}} = \dot{\alpha}\left(\frac{27z^2}{4\mu}e^{-3\alpha} - 12ze^{-2\alpha}\right) + \dot{z}\left(6e^{-2\alpha} - \frac{9z}{2\mu}e^{-3\alpha}\right) = 0 \quad . \tag{3.38}$$

Equation (3.38) leads to

$$\frac{dz}{z} = \frac{\left(4e^{\alpha} - \frac{9z}{4\mu}\right)}{\left(2e^{\alpha} - \frac{3z}{2\mu}\right)}d\alpha \quad . \tag{3.39}$$

Thus, defining

$$x \equiv e^{-\alpha} \quad , \tag{3.40}$$

$$xz \equiv \eta(\xi(x)) \quad , \tag{3.41}$$

$$\xi(x) \equiv \log(x) \quad , \tag{3.42}$$

$$f(\eta) \equiv -\frac{\left(4 - \frac{9\eta}{4\mu}\right)}{\left(2 - \frac{3\eta}{2\mu}\right)} \quad , \tag{3.43}$$

differentiation of (3.41) with respect to x and comparison with (3.39) leads to the differential equation

$$\frac{d\eta}{d\xi} = \eta\Big(1 + f(\eta)\Big) \quad . \tag{3.44}$$

The corresponding solution for x may be expressed in the form (see Eq. I.I26 on page 316 of [20])

$$\log(x) = \int_C^{xz} \frac{d\eta}{\eta(1+f(\eta))} \quad . \tag{3.45}$$

By virtue of (3.43), Eq. (3.45) implies, after performing some standard integrals, that

$$x^2 z \left(\frac{3}{4\mu} xz - 2 \right) = \widetilde{A} \quad , \tag{3.46}$$

where $\widetilde{A}$ is an integration constant. Thus, by using the definition (3.40) and the relation between z and the scalar curvature: $z = \frac{\mu}{3} e^{3\alpha} R$, (3.46) leads to the second-order algebraic equation for the scalar curvature

$$a^3 R^2 - 8aR - 4\widetilde{A} = 0 \quad . \tag{3.47}$$

Although it seems impossible to solve by analytic methods the field equations in Hamiltonian (cf. [2]) or Lagrangian form, we have been able to re-express first- and second-generation constraints in terms of physically relevant quantities, i.e. the cosmic scale factor and the scalar curvature. Interestingly, the roots of (3.47) are given by

$$R = \frac{4}{a^2} \pm \frac{2}{a^2} \sqrt{4 + \widetilde{A} a} \quad . \tag{3.48}$$

Thus, if second-order formalism is used, the scalar curvature does not vanish in vacuum in the presence of torsion, unless the constant $\widetilde{A}$ is set to zero and the negative sign is chosen in front of the square root. Moreover, if the constant $\widetilde{A}$ is negative, the cosmic scale factor is bound to remain less than or equal to $\frac{4}{|\widetilde{A}|}$.

By contrast, if torsion vanishes, our model corresponds to a Hamiltonian system with first-class constraints (cf. [7]). At a Lagrangian level, the first-generation constraint is given by E_L/N, where E_L is the energy function and N is the lapse function. Thus, such a constraint vanishes if and only if the energy function vanishes, and is automatically preserved, since E_L is constant along solutions of the field equations (hence its Lie derivative along the vector field solving the Lagrange equations vanishes).

4. - Concluding remarks.

The contribution of this paper is a detailed application of the generalized method of section 3. Since it relies on an approach [17-18] whose range of applicability is wider than any previous (Lagrangian) method, we found it appropriate to focus on this technique in our paper. Remarkably, in our specific model, if Lagrange equations are studied in second-order formalism, a second-generation constraint is found which is absent in first-order formalism. This constraint has been expressed in terms of the physical quantities of the problem, as shown in Eq. (3.38). One thus finds the equation (3.47) for the scalar curvature, whose roots are given by (3.48). It turns out that the scalar curvature may not vanish in vacuum. Interestingly, an upper limit for the cosmic scale factor exists if the constant $\widetilde{A}$ in (3.48) is negative. Hence second-order theory is found to yield relevant information about the early universe in our toy model.

Note also that the secondary constraint of equation (2.10) is obtained multiplying by $-\frac{1}{2}$ the secondary constraint of equation (1.7), as one may check by using (2.1) and the map

$$FL(q^i, v^i) \equiv \left(q^i, p_i \equiv \frac{\partial L}{\partial v^i}\right) \quad , \tag{4.1}$$

which expresses the map in local coordinates from the tangent bundle TQ to the cotangent bundle T^*Q. This implies that, up to an unessential proportionality constant, the Hamiltonian and the Lagrangian method lead to the same secondary constraint. However, the careful reader may have noticed that, using tangent-bundle formalism, no primary constraint occurs. Such property is not surprising for the following reasons. If det $\mid \frac{\partial^2 L}{\partial v^i \partial v^j} \mid = 0$, the map FL defined in (4.1) maps the tangent bundle TQ to the primary-constraint submanifold $\widehat{\Sigma}$ of the phase space T^*Q. Thus, $(FL)^*$ maps (by definition of pull-back) forms on $\widehat{\Sigma}$ to forms on TQ, and the forms ω_L, dE_L appearing in the Lagrange equation (2.11) correspond to geometrical objects *already* living on $\widehat{\Sigma} \subset T^*Q$. Interestingly, the first set of constraints one actually evaluates in the Lagrangian formalism are the equivalent of secondary constraints of Dirac's theory. This property appears very important for first-class theories such as Maxwell's electrodynamics and GR: the Lagrangian method is a powerful way to evaluate directly the constraints which govern the theory, i.e. first-class secondary constraints. Although the question remains open [11-13,21], Dirac's consideration of first-class secondary constraints appears more natural within the Lagrangian approach.

A naturally occurring question is whether a Lagrangian multisymplectic analysis [22] of first-class theories such as Einstein's GR offers any advantage with respect to the ADM Hamiltonian method [13,23-24] which relies instead on the 3+1 split of the space-time geometry, especially in light of the longstanding unsolved problems of the quantization program [13]. Work is now in progress by the authors on some of these issues, and we hope that the geometrical reformulation of certain properties of classical field theory presented in this paper will serve as a first step towards this more ambitious goal.

* * *

G. Esposito is grateful to Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, and to Professor Dennis Sciama for hospitality at the SISSA of Trieste, during the early stages of this work. G. Marmo is indebted to Professor Ibort for hospitality at the Departamento de Fisica Teorica of the Universidad Complutense of Madrid. Conversations with Giovanna Mendella and correspondence with Claes Uggla are also gratefully acknowledged.

REFERENCES

[1] G. MORANDI, C. FERRARIO, G. LO VECCHIO, G. MARMO and C. RUBANO: *Phys. Reports*, **188**, 147 (1990).

[2] G. ESPOSITO: *Nuovo Cimento B*, **104**, 199 (1989).

[3] E. SEZGIN and P. van NIEUWENHUIZEN: *Phys. Rev. D*, **21**, 3269 (1980).

[4] I. A. NIKOLIC: *Phys. Rev. D*, **30**, 2508 (1984).

[5] V. SZCZYRBA: *Phys. Rev. D*, **36**, 351 (1987).

[6] V. SZCZYRBA: *J. Math. Phys.* (N.Y.), **28**, 146 (1987).

[7] G. T. HOROWITZ: *Phys. Rev. D*, **31**, 1169 (1985).

[8] G. ESPOSITO: *Nuovo Cimento B*, **106**, 1315 (1991).

[9] M. A. H. MACCALLUM: *Anisotropic and inhomogeneous relativistic cosmologies*, in *General Relativity, an Einstein Centenary Survey*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1979).

[10] D. BOULWARE: *Quantization of higher-derivative theories of gravity*, in *Quantum Theory of Gravity*, edited by S. M. Christensen (Adam Hilger, Bristol, 1984).

[11] P. A. M. DIRAC: *Lectures on Quantum Mechanics*, Belfer Graduate School of Science (Yeshiva University, New York, 1964).

[12] G. MARMO, N. MUKUNDA and J. SAMUEL: *Riv. Nuovo Cimento*, **6**, 1 (1983).

[13] G. ESPOSITO: *Quantum Gravity, Quantum Cosmology and Lorentzian Geometries*, second corrected and enlarged edition, Lecture Notes in Physics, New Series m: Monographs, Volume m12 (Springer-Verlag, Berlin, 1994).

[14] M. J. GOTAY and J. M. NESTER: *Ann. Inst. H. Poincaré A*, **30**, 129 (1979).

[15] J. F. CARINENA, C. LOPEZ and N. ROMAN-ROY: *J. Geom. Phys.*, **4**, 315 (1987).

[16] G. MARMO, E. J. SALETAN, A. SIMONI and B. VITALE: *Dynamical Systems, a Differential Geometric Approach to Symmetry and Reduction* (John Wiley, New York, 1985).

[17] G. MENDELLA: *Geometrical Aspects of Lagrangian Dynamical Systems with Dirac Constraints*, Physics-Degree Thesis (University of Napoli, 1987); *A Geometrical Formalism for Constraints' Theory*, Ph.D. Thesis (University of Napoli, 1992).

[18] G. MARMO, G. MENDELLA and W. M. TULCZYJEW: *Integrability of Implicit Differential Equations*, paper in preparation (DSF preprint 93/51).

[19] K. SUNDERMEYER: *Constrained Dynamics* (Springer-Verlag, Berlin, 1982).

[20] E. KAMKE: *Differentialgleichungen Lösungsmethoden und Lösungen* (Chelsea Publishing Company, New York, 1959).

[21] M. E. V. COSTA, H. D. GIROTTI and T. J. M. SIMOES: *Phys. Rev. D*, **32**, 405 (1985).

[22] M. GOTAY, J. ISENBERG, J. MARSDEN, R. MONTGOMERY, J. SNIATYCKI and P. B. YASSKIN: *Momentum Maps and the Hamiltonian Treatment of Classical*

Field Theories with Constraints, to appear in Mathematical Sciences Research Institute Publications (Springer-Verlag, Berlin).

[23] A. ASHTEKAR: *Lectures on Non-Perturbative Canonical Gravity* (World Scientific, Singapore, 1991).

[24] M. FERRARIS, M. FRANCAVIGLIA and I. SINICCO: *Nuovo Cimento B*, **107**, 1303 (1992).